Pergamon

J. Appl. Maths Mechs, Vol. 59, No. 5, pp. 805–816, 1995

0021–8928/95 $24.00+0.00

0021–8928(95)00095-X

THE GENERAL NON-INERTIAL INITIAL-BOUNDARY-VALUE PROBLEM FOR A VISCOELASTIC AGEING SOLID WITH PIECEWISE-CONTINUOUS ACCRETION†

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(*Received* 26 *May* 1994)

The deformation processes in a viscoelastic body whose composition, mass or volume varies in a piecewise-continuous manner due to the addition of new material to the outer surface of the body is studied. Modelling of such processes leads to basically new non-classical problems in solid mechanics. The formulation is considered and a method of constructing the solution of the general non-inertial initial-boundary-value problem for a viscoelastic ageing body with piecewise-continuous accretion is suggested. The fundamental theorems are stated and some qualitative features of the evolution of the stress–strain state of the growing body are studied.

The practical significance and future development of growing body mechanics is determined by the fact that practically all objects in solid mechanics (buildings, structures, structural components, machine parts, etc.) originate as a result of the construction, solidification, deposition by spraying, building up, deposition by freezing, winding, etc. Processes involving the continuous erection of concrete structures, metal solidification, spray deposition of semiconducting films, crystal growth, etc. are specific examples of processes of this kind.

1. CHARACTERISTIC FEATURES OF THE BASIC RELATIONSHIPS IN THE MECHANICS OF CONTINUOUSLY GROWING BODIES AND FORMULATION OF THE PROBLEM

The present section contains a number of general definitions, assumptions and relations in growing-body mechanics. Although based on [1–3] and the monographs [4, 5], it is not a combination of these, but is a brief independent account of the basics with a view to an effective formulation and formulation and solution of a more complex problem.

By a (piecewise) continuously *growing* body we mean a solid body whose composition, mass or volume varies as a result of a (piecewise) continuous addition of material to its surface (see Fig. 1). The process of adding new material to the body is called *accretion* or *growth*.

For piecewise-continuous accretion the following *basic stages* of its deformation are strictly followed: before accretion, during the continuous growth process a deformation stage alternating with the latter, and after the accretion has ceased and growth has stopped. Each of these stages is characterized by the times when it starts and ends. The first is characterized by the time of application of a load to the body and the time when growth starts. The second by the time when growth starts and the time when it ends. Conversely, the third is characterized by the time when growth ends and the time when it starts. The process under investigation is usually concluded by the third stage, for which the time when the next stage begins is taken to be as long as desired.

The body on whose surface new material is deposited starting from the time when accretion starts is called the *basic* or *original* body. The body consisting of the material pieces added to the basic body over the time interval from the beginning of accretion up to a given instant of time is called the *additional* body. The additional body can have a complex structure and consist of a collection of bodies formed over different time intervals of continuous accretion. We call them *sub-bodies*. The additional body is obviously the union of sub-bodies. The domains occupied by the former and latter can be disconnected. The union of the basic and the additional bodies will be called the *accreted* or *growing* body. Note that accretion can also occur without the basic body, starting from an infinitesimal material element.

†*Prikl. Mat. Mekh.* Vol. 59, No. 5, pp. 836–848, 1995.

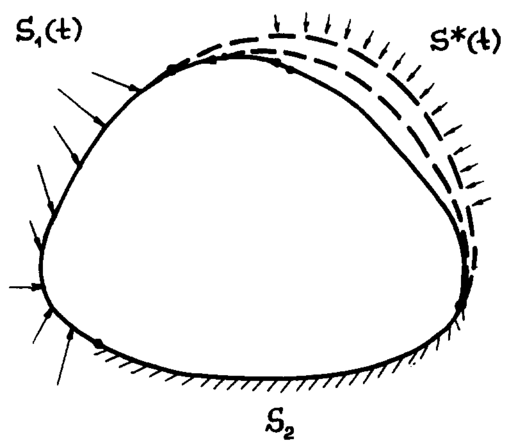

Fig. 1.

The part of the surface where infinitesimal pieces of the material are deposited at the actual instant is called the *accretion* or *growth* surface. The growth surface may be disconnected, in general. In particular, it can be the whole surface of the body. Finally, the part of the surface of the original or the growing body that coincides with the growth surface at the time when growth starts will be called the *base* surface. The base surface is clearly the part of the surface of the body on which material is to be deposited during the next stage of continuous accretion. At different stages it coincides, as a rule, with the surface between the basic body and the additional body as well as with the surfaces between the sub-bodies.

We shall assume that the basic body, which is made from a viscoelastic ageing material, occupies a domain Ω_0 with the surface S_0 and is free of stresses up to the time τ_0 of application of the load. From τ_0 up to the time τ_1 when accretion starts the classical boundary conditions are given on S_0, the specific form of which is stated below. At τ_1 the continuous accretion of a solid begins due to the addition of material particles to the accretion surface $S^*(t)$. As it grows, the body occupies a domain $\Omega(t)$ with surface $S(t)$. It is obvious that $S^*(t) \subseteq S(t)$.

The time when a particle characterized by a position vector x is deposited on the body will be denoted by $\tau^*(x)$ and called the time of deposition of the particle on the growing body. The configuration of the accreted body is completely defined by the function $\tau^*(x)$ depending on the spatial coordinates. Boundedness and piecewise-continuity are the general conditions usually imposed on $\tau^*(x)$.

We will denote by $\tau_1^*(x)$ the time when an element of the growing body is formed and by $\tau_0(x)$ the time when a load is applied to it. Naturally, $\tau_1^*(x) \leqslant \tau_0(x) = \tau_0$ for the elements of the basic body ($x \in \Omega_0$).

The vector equilibrium equation is obviously satisfied in the domain occupied by the growing body at each instant of time. For quasistatic processes it has the form

$$x \in \Omega(t): \ \nabla \cdot T + f = 0 \tag{1.1}$$

where T is the stress tensor, f is the vector of volume forces, and ∇ is the Hamilton operator (here and henceforth we use the notation of tensor calculus from [6]).

The Cauchy conditions and the compatibility equations for deformations are always satisfied in the domain occupied by the basic body

$$x \in \Omega_0: \ E = \tfrac{1}{2}[\nabla u + (\nabla u)^T], \quad \nabla \times (\nabla \times E)^T = 0 \tag{1.2}$$

where E is the strain tensor and u is the displacement vector. But in the domain $\Omega^*(t)$ occupied by the additional body ($\Omega^*(t) = \Omega(t)/\Omega_0$) only their analogues involving the rates of change of the corresponding variables are satisfied

$$x \in \Omega^*(t): \ D = \tfrac{1}{2}[\nabla \upsilon + (\nabla \upsilon)^T], \quad \nabla \times (\nabla \times D)^T = 0 \tag{1.3}$$

$$D = \partial E/\partial t, \quad \upsilon = \partial u/\partial t$$

i.e. the strains are incompatible, in general.

The latter reflects the fact that prior to the time of deposition on the basic body the deposited elements may be subject to deforming actions independently of the processes taking place in the body itself.

To study the stress–strain state (SSS) of the growing body one must know the laws of deformation of the basic body from the time τ_0 when the load is applied up to the time τ_1 when accretion starts and of the deposited material from the time $\tau_0(x)$ when a load is applied to this material up to the time $\tau^*(x)$ of their deposition on the growing body. If the state of the original body is determined from the solution of the problem with fixed boundary, then the previous history of the strain tensor of infinitesimally thin continuously deposited material is assumed to be known

$$x \in \Omega^*(t),\ \ \tau_0(x) \leqslant \tau \leqslant \tau^*(x):\ E(x,\tau) = E_0(x,\tau) \tag{1.4}$$

The prescribed history of the strain tensor of the deposited material provides a specific *initial-boundary* condition at the time of deposition on the growing surface $S^*(t)$

$$x \in S^*(t):\ E(x,\tau^*(x)) = E^*(x),\ \ t = \tau^*(x) \tag{1.5}$$

In particular, by the constitutive equations, the complete stress tensor $T^*(x)$ is, as a rule, also defined on the growth surface. It is consistent with the external loads and characterizes the pretension of the deposited elements.

Furthermore, we observe that

$$\tau^*(x) = t \tag{1.6}$$

is the equation of the growth surface and, by (1.6)

$$s_n = |\nabla\tau^*(x)|^{-1} \tag{1.7}$$

is the velocity of motion of the surface $S^*(t)$ in the normal direction.

The traditional boundary conditions for the displacement vector and the vector of surface forces are given on the stationary sections of the surface of the growing body.

To describe the behaviour of the material of the growing body we will use the constitutive equations for an inhomogeneous ageing body (see [7]). Extending the definition of $\tau_0(x)$ by a constant τ_0 to the domain occupied by the original body, we write

$$\operatorname{dev} E(x,t) = \frac{\operatorname{dev} T(x,t)}{2G(t-\tau_1^*(x),x)} - \int_{\tau_0(x)}^{t} \frac{\operatorname{dev} T(x,\tau)}{2G(\tau-\tau_1^*(x),x)} K_1(t-\tau_1^*(x),\tau-\tau_1^*(x),x)\,d\tau \tag{1.8}$$

$$I_1[E(x,t)] = \frac{I_1[T(x,t)]}{k^*(t-\tau_1^*(x),x)} - \int_{\tau_0(x)}^{t} \frac{I_1[T(x,\tau)]}{k^*(\tau-\tau_1^*(x),x)} K_2(t-\tau_1^*(x),\tau-\tau_1^*(x),x)\,d\tau \tag{1.9}$$

where $G(t)$, $k^*(t)$ and $K_1(t,\tau)$, $K_2(t,\tau)$ are the instantaneous elastic strain moduli and creep functions for pure shear and uniform compression, respectively, $I_1(M)$ denotes the first invariant of a tensor M, and $\operatorname{dev} M$ is the deviator of M.

The description of the process of continuous accretion of a viscoelastic ageing body involves three characteristic times: the time when the element with coordinate $x - \tau_1^*(x)$ is formed, the time $\tau_0(x)$ when a load is applied to this element, and the time $\tau^*(x)$ when the element is deposited on the growing body. These three times are different, in general.

The deposition process is largely determined by specifying these three times. If the processes of continuous concrete casting, ice formation, crystal growth, etc. are studied, then $\tau_1^*(x) = \tau_0(x) = \tau^*(x)$, i.e. the elements are deposited at the same time as they are formed and a load is applied to them. If spray deposition or erection of a structure from a large number of blocks is modelled by a continuous growth process, then, as a rule, $\tau_0(x) = \tau^*(x)$ and the time $\tau_1^*(x)$ when the elements are formed is arbitrary. If the deformation of elements begins as soon as they are formed and they are being added to the basic body only over some time interval, then $\tau_1^*(x) = \tau_0(x) \neq \tau^*(x)$ and so on.

Before formulating the problem considered in the present paper, we emphasize that the problem of the growth of a body differs in a major way from that involving the removal of material. The latter is characterized solely by the fact that the domain occupied by the body is reduced, subject to the standard equations and boundary conditions.

Suppose that a homogeneous viscoelastic ageing body occupying a domain Ω_0 with surface $S_0 (x \in \Omega_0)$ is formed at time $\tau_1^*(x) = 0$ and is free of stresses up to the time $\tau_0 \geqslant 0$ when a load is applied. Starting from the latter time, in general four kinds of boundary conditions can be given on the surface of the body (surface forces on $S_1(t)$, displacements on $S_2(t)$, normal displacements and tangential forces on $S_3(t)$, and normal forces and tangential displacements on $S_4(t)$) in addition to volume forces, which depend on the coordinates and time.

The sections of the surface on which different boundary conditions are given do not intersect one another and cover the whole surface of the body. The dependence of S_i on t enables us to take into account the possible evolution of the system of loads, punches, etc. on S_0, and is assumed to be piecewise-constant. Unless the body surface is closed, the behaviour of stresses or strains at infinity is prescribed.

Continuous deposition of material formed simultaneously with the body ($\tau_1^*(x) = 0$) starts at $\tau_1 \geqslant \tau_0$. The body occupies a domain $\Omega(t)$ with surface $S(t)$ during its growth. The growth surface $S^*(t)$ ($S^*(\tau_1) \subset S_0$) moves in space. The sections $S_i(t)$ ($i = 1, \ldots, 4$) on which the common boundary conditions are given can vary because of the loading of the stationary surface of the additional body. We shall assume that the complete stress tensor defined on the growth surface is consistent with the known surface forces $p^*(x, t)$ on $S^*(t)$ (for example, with pressure) and the time when a load is applied to the deposited material coincides with the time when they become deposited on the growing body ($\tau_0(x) = \tau^*(x)$).

At the instant $\tau_2 > \tau_1$ the accretion of the body ceases, and starting from this instant four kinds of boundary conditions are given on the sections $S_i(t)$ of the surface $S_1 = S(\tau_2)$ of the body occupying the domain $\Omega_1 = \Omega(\tau_2)$.

After some time, at the instant $\tau_3 > \tau_2$ the body accretion may start again. An accretion surface may appear which is not related in any way to the previous one. Then the accretion may stop at time τ_4, and so on, leading to the problem of piecewise-continuous accretion of a solid with n times at which the growth starts and, respectively, n times when it stops.

Proceeding to the study of the basic stages of the process of piecewise-continuous accretion of a viscoelastic body, we note that fairly slow processes will be considered everywhere below, so that the inertial terms can be neglected in the equilibrium equations. Furthermore, to avoid complex expressions, only the first two kinds of conditions are given on the body surface. Their subsequent transformations yield the precise scheme of this investigation by analogy with the boundary conditions of the third and fourth kind.

2. THE STRESS-STRAIN STATE OF THE BASIC BODY PRIOR TO THE ACCRETION PROCESS

We consider the SSS of a viscoelastic ageing body Ω_0 in the time interval $[\tau_0, \tau_1]$. We write the equilibrium equation in the form (1.1)

$$\nabla \cdot T + f = 0 \tag{2.1}$$

We represent the boundary conditions described above as follows:

$$x \in S_1(t):\ n \cdot T = p_0, \quad x \in S_2(t):\ u = u_0 \tag{2.2}$$

where p_i and u_i are given vectors of surface forces and strains and n is the unit vector normal to the body surface. The Cauchy conditions are written as follows (see (1.2)):

$$E = \tfrac{1}{2}[\nabla u + (\nabla u)^T] \tag{2.3}$$

We take the constitutive equations in the form (1.8) and (1.9), assuming that the transverse contraction ratio of the instantaneous elastic strain and the creep strain of the ageing material are identical and are equal to ν. Then we have (see [5, 8])

$$\begin{gathered} T = G(I + N(\tau_0, t))\,[2E + (K-1)\,I_1(E)\,1] \\ (I + N(\tau_0^{\prime}, t))^{-1} = (I - L(\tau_0, t)) \\ G = \frac{E_1}{2(1+\nu)}, \quad K = \frac{1}{(1-2\nu)}, \quad C(t,\tau) = \frac{\omega(t,\tau)}{2(1+\nu)}, \quad L(\tau_0, t)\, f(t) = \int_{\tau_0}^{t} f(\tau)\, K(t,\tau)\, d\tau \\ K(t,\tau) = E_1(\tau)\frac{\partial}{\partial \tau}[E_1^{-1}(\tau) + C(t,\tau)] = K_1(t,\tau) = G(\tau)\frac{\partial}{\partial \tau}[G^{-1}(\tau) + \omega(t,\tau)] \end{gathered} \tag{2.4}$$

where $E_1 = E_1(t)$ and $G = G(t)$ are the instantaneous strain moduli under tension and shear, $C(t, \tau)$ and $\omega(t, \tau)$ are the creep measures under tension and shear, $K(t, \tau)$ is the creep function under tension, and $\mathbf{1}$ is the unit tensor. The arguments are omitted in a number of obvious cases above. They will also be omitted in what follows and will be used only in those cases when their absence may be misleading.

Thus (2.1)–(2.4) constitute the boundary-value problem (BVP) of the linear theory of elasticity for a homogeneous ageing basic body, the SSS of which can be described by the solution of the system for $t \in [\tau_0, \tau_1]$.

We shall transform the BVP for the basic body. Let us introduce the notation

$$M^0 = H(\tau_0, t) MG^{-1}, \quad a^0 = H(\tau_0, t) aG^{-1}, \quad H(\psi, t) = (I - L(\psi, t)) \tag{2.5}$$

where M and a are an arbitrary tensor and arbitrary vector, respectively. We apply the operator $H(\tau_0, t)$ to the relations in (2.1)–(2.4) containing T after dividing them by G. Then, since $H(\tau_0, t)$ commutes with the Hamilton operator, we obtain the following BVP using (2.5) ($\tau_0 \leqslant t \leqslant \tau_1$)

$$\nabla \cdot T^0 + f^0 = 0, \quad x \in S_1(t): \; n \cdot T^0 = p_0^0, \quad x \in S_2(t): \; u = u_0 \tag{2.6}$$

$$E = \tfrac{1}{2}[\nabla u + (\nabla u)^T], \quad T^0 = 2E + (K-1) I_1(E) \mathbf{1}$$

Unlike (2.1)–(2.4), time occurs in the BVP (2.6) as a parameter. The latter is mathematically equivalent to the BVP of the theory of elasticity with a parameter t. All analytic and numerical methods of the theory of elasticity can be used when constructing the solution of such a problem, which undoubtedly lends itself better to investigation than the problem (2.1)–(2.4) of the theory of viscoelasticity. However, these advantages of the BVP (2.6) can only be fully utilized when there is a one-to-one correspondence between the solutions (2.1)–(2.4) and (2.6). This is provided by the following theorem.

Theorem 1. in order that T, E and u be a solution of the BVP (2.1)–(2.4) it is necessary and sufficient that T^0, E, and u form a solution of the BVP (2.6) and the relation ($\tau_0 \leqslant t \leqslant \tau_1$)

$$T(x,t) = G(t)[T^0(x,t) + \int_{\tau_0}^{t} T^0(x,\tau) R(t,\tau) d\tau] \tag{2.7}$$

be satisfied. Here $R(t, \tau)$ is the resolvent of the kernel $K(t, \tau)$.

The theorem can be easily verified directly, using the fact that the homogeneous Volterra equation of the second kind has only a trivial solution (see, for example, [9, 10]). Here we just observe that (2.5) and (2.7) provide examples of such equations written in the direct and resolvent forms, respectively.

Therefore, solving the BVP (2.6) with t as a parameter, one can reconstruct the true characteristics of the SSS of the original viscoelastic ageing body (2.1)–(2.4) from Theorem 1 using (2.7) (see also the correspondence principle [8]).

The transformations below have largely a formal character and will be used in the following sections to simplify and unify as much as possible the basic computational formulae for the piecewise-continuous accretion process.

We differentiate (2.6) with respect to t (here and henceforth differentiation is understood in the sense of distributions). Then we have

$$\nabla \cdot S + h = 0, \quad x \in S_1(t): \; n \cdot S = w_0, \quad x \in S_2(t): \; \upsilon = \upsilon_0 \tag{2.8}$$

$$D = \tfrac{1}{2}[\nabla \upsilon + (\nabla \upsilon)^T], \quad S = 2D + (K-1) I_1(D) \mathbf{1}$$

$$S = \frac{\partial T^0}{\partial t}, \quad \upsilon_0 = \frac{\partial u_0}{\partial t}, \quad h = \frac{\partial f^0}{\partial t}, \quad w_0 = \frac{\partial p_0^0}{\partial t}$$

$$w_0 = \frac{1}{G(t)} \frac{\partial p_0(x,t)}{\partial t} + \int_{\tau_0}^{t} \frac{\partial p_0(x,\tau)}{\partial \tau} \frac{\partial \omega(t,\tau)}{\partial t} d\tau + p_0(x,\tau_0) \frac{\partial \omega(t,\tau)}{\partial t} \tag{2.9}$$

It is obvious that relations (2.8), supplemented by the initial conditions for the differentiated functions, form a so-called initial-boundary-value problem (IBVP) (see, for example, [11]), which depends on t as a parameter. The initial conditions themselves can be found from the solution of (2.6) by setting $t = \tau_0$.

Theorem 2. The functions T^0, E and u form a solution of the BVP (2.6) if and only if S, D and υ satisfy the IBVP (2.8) and ($\tau_0 \leqslant t \leqslant \tau_1$)

$$T^0(x,t) = T^0(x,\tau_0) + \int_{\tau_0}^{t} S(x,\tau)\, d\tau, \quad E(x,t) = E(x,\tau_0) + \int_{\tau_0}^{t} D(x,\tau)\, d\tau \tag{2.10}$$

$$u(x,t) = u(x,\tau_0) + \int_{\tau_0}^{t} \upsilon(x,\tau)\, d\tau$$

Theorems 1 and 2 enable one to state the following assertion.

Theorem 3. For functions T, E and u to be a solution of the BVP (2.1)–(2.4) it is necessary and sufficient that S, D and υ form a solution of (2.8) and (see (2.7), (2.10))

$$T(x,t) = G(t)\left\{ \frac{T(x,\tau_0)}{G(\tau_0)} \left[1 + \int_{\tau_0}^{t} R(t,\tau)\, d\tau \right] + \int_{\tau_0}^{t} \left[S(x,\tau) + \int_{\tau_0}^{\tau} S(x,\zeta)\, d\zeta\, R(t,\tau) \right] d\tau \right\}$$

$$u(x,t) = u(x,\tau_0) + \int_{\tau_0}^{t} \upsilon(x,\tau)\, d\tau \quad (x \in \Omega_0, \ t \in [\tau_0, \tau_1]) \tag{2.11}$$

We observe that the BVP based on (2.8) has the same form as the BVP of the theory of elasticity with parameter t. To solve (2.8) one can use all the methods (analytic and numerical) known in the theory of elasticity. Once (2.8) is solved, the true characteristics of the SSS of the basic body can be reconstructed from (2.11).

3. THE INITIAL-BOUNDARY-VALUE PROBLEM FOR A CONTINUOUSLY GROWING BODY

We shall now consider the process of continuous accretion of a solid ($\tau_1 \leqslant t \leqslant \tau_2$). For a growing body we have:
the equilibrium equation

$$\nabla \cdot T + f = 0 \tag{3.1}$$

the boundary conditions on the stationary part of the surface

$$x \in S_1(t):\ n \cdot T = p_0, \quad x \in S_2(t):\ u = u_0 \tag{3.2}$$

the initial-boundary condition on the growing surface

$$x \in S^*(t):\ T = T^*, \quad n \cdot T^* = p^* \quad (t = \tau^*(x)) \tag{3.3}$$

the relation between the rates of strain and displacement

$$D = \tfrac{1}{2}[\nabla \upsilon + (\nabla \upsilon)^T] \tag{3.4}$$

and the constitutive equation of the form

$$T = G(I + N(\tau_0(x), t))\,[2E + (K-1)\, I_1(E)\mathbf{1}] \tag{3.5}$$

$$\tau_0(x) = \begin{cases} \tau_0, & x \in \Omega_0 \\ \tau^*(x), & x \in \Omega^*(t) \end{cases}$$

Relations (3.1)–(3.5) form a general non-inertial IBVP for a continuous growing body, where $T^* = T(x, \tau^*(x))$ is the complete stress tensor given on $S^*(t)$, which is consistent with the external forces p^*, and where the operator $(I - L(\tau_0(x), t)) = H(\tau_0(x), t))$ and its inverse $(I + N(\tau_0(x), t))$ can be determined from (2.4) and (2.5) with τ_0 replaced by $\tau_0(x)$. We observe (see (3.5)) that the process of continuous deposition of new elements on the basic body under investigation leads, in general, to governing relations containing discontinuities on the interface between the original and the additional bodies.

We transform the IBVP for a continuously accreted viscoelastic ageing body into a problem with the time as a parameter that has the same form as the BVP of the theory of elasticity (see also [1–5]). At the first stage we transform the growth problem for a viscoelastic body with the constitutive equations (3.5) into the problem of accretion of an elastic body described by Hooke's law.

We represent the equation of the base surface $S_* = S^*(\tau_1)$ in the form (see (1.6))

$$x \in S_*: \ \tau^*(x)\,\tau_1^{-1} - 1 = 0 \tag{3.6}$$

where $\tau^*(x)\tau_1^{-1} - 1 < 0$ for $x \in \Omega_0$ and $\tau^*(x)\tau_1^{-1} - 1 \geqslant 0$ for $x \in \Omega^*$. We shall assume that $\tau^*(x)$ is a sufficiently smooth function such that $\nabla\tau^*(x) \neq 0$ when $\tau^*(x)\tau_1^{-1} - 1 = 0$ (i.e. there are no singular points on S_*). We introduce the characteristic function $\theta(\tau^*(x)\tau_1^{-1} - 1)$, which is equal to one when its argument is greater than or equal to zero, and equal to zero when the argument is negative [12]. Now $H(\tau_0(x), t)$ can be represented in the form

$$H(\tau_0(x),t)\,\varphi(t) = (I - L(\tau^0(x),t))\,\varphi(t) - [1-\theta(\tau^*(x)\,\tau_1^{-1}-1)]\,L^t(\tau_0,\tau_1)\,\varphi(t) \tag{3.7}$$

$$L^t(\tau_0,\tau_1)\,f(t) = \int_{\tau_0}^{\tau_1} f(\tau)\,K(t,\tau)\,d\tau, \quad \tau^0(x) = \tau_1 + \theta(\tau^*(x)\,\tau_1^{-1}-1)\,(\tau^*(x)-\tau_1) \tag{3.8}$$

where $\tau^*(x) = \tau_1$ for $x \in S_*$ (see (3.6)).

We apply the operator $H(\tau_0(x), t)$ to the relations in (3.1)–(3.5) containing T after dividing them by G. Then

$$\begin{gathered} H\nabla\cdot TG^{-1} + f^0 = 0, \quad x \in S_1(t):\ n\cdot T^0 = p_0, \quad x\in S_2(t):\ u = u_0 \\ x \in S^*(t):\ T^0 = T^{*0} = T^*(x)\,G^{-1}(\tau^*(x)) \\ n\cdot T^{*0} = p^{*0} = p^*(x)\,G^{-1}(\tau^*(x)) \quad (t = \tau^*(x)) \\ D = \tfrac{1}{2}[\nabla\upsilon + (\nabla\upsilon)^T], \quad T^0 = 2E + (K-1)\,I_1(E)\,\mathbf{1} \end{gathered} \tag{3.9}$$

Now, using (3.7)–(3.8), we transform (3.9) into an IBVP relative to T^0, E and u, where t plays the role of a parameter.

It can be shown that (see (1.6), (1.7) and (3.6)–(3.8))

$$\begin{gathered} H\nabla\cdot TG^{-1} = \nabla\cdot T^0 - \theta(\tau^*(x)\,\tau_1^{-1}-1)\,s_n^{-1} n\cdot T^*(x)\,G^{-1}(\tau^*(x))\,K(t,\tau^*(x)) - \\ -\delta(\tau^*(x)\,\tau_1^{-1}-1)\,\tau_1^{-1}s_n^{-1}\int_{\tau_0}^{\tau_1} n\cdot T(x,\tau)\,G^{-1}(\tau)\,K(t,\tau)\,d\tau \\ \nabla\tau^0(x) = \theta(\tau^*(x)\,\tau_1^{-1}-1)\,\nabla\tau^*(x), \quad \nabla\tau^*(x) = s_n^{-1} n \end{gathered} \tag{3.10}$$

The second term on the right-hand side of (3.10) is known from the conditions of the problem (see (3.9)) and the third term can be determined from the solution of the problem for the basic body for $t \in [\tau_0, \tau_1]$. Setting

$$x \in S_*:\ n\cdot T(x,\tau) = p_*(x,\tau), \quad p_*^0(x,\tau) = p_*(x,\tau)\,G^{-1}(\tau) \quad (\tau_0 \leqslant \tau \leqslant \tau_1) \tag{3.11}$$

where $p_*(x, \tau)$ is the load acting on the base surface of the basic body up to the time when growth starts, we obtain

$$\begin{gathered} H\nabla\cdot TG^{-1} = \nabla\cdot T^0 - \theta(\tau^*(x)\,\tau_1^{-1}-1)\,f_1^0(x,t) - \delta(\tau^*(x)\,\tau_1^{-1}-1)\,f_2^0(x,t) \\ f_1^0(x,t) = s_n^{-1}p^{*0}(x)\,K(t,\tau^*(x)), \quad f_2^0(x,t) = s_n^{-1}\tau_1^{-1}\int_{\tau_0}^{\tau_1} p_*^0(x,\tau)\,K(t,\tau)\,d\tau \end{gathered} \tag{3.12}$$

where $f_1^0(x, t)$ and $f_2^0(x, t)$ are known vector functions.

Relations (3.12) enable us to write (3.9) in the form

$$
\begin{aligned}
&\nabla \cdot T^0 + f^0 - \theta(\tau^*(x)\,\tau_1^{-1} - 1)\, f_1^0(x,t) - \delta(\tau^*(x)\,\tau_1^{-1} - 1)\, f_2^0(x,t) = 0 \\
&x \in S_1(t):\ n \cdot T^0 = p_0^0, \quad x \in S_2(t):\ u = u_0 \\
&x \in S^*(t):\ T^0 = T^{*0}, \quad n \cdot T^{*0} = p^{*0} \quad (t = \tau^*(x)) \\
&D = \tfrac{1}{2}[\nabla \upsilon + (\nabla \upsilon)^T], \quad T^0 = 2E + (K-1)\, I_1(E)\,\mathbf{1}
\end{aligned}
\tag{3.13}
$$

Theorem 4. For T, E and u to be solutions of the IBVP (3.1)–(3.5) it is necessary and sufficient that T^0, E and u form a solution of the IBVP (3.13) and that the following relation ($t \geqslant \tau_1$) is satisfied

$$
T(x,t) = G(t)\,[T^0(x,t) + \int_{\tau_0(x)}^{t} T^0(x,\tau)\, R(t,\tau)\, d\tau]
\tag{3.14}
$$

By analogy with Theorem 1, the proof of Theorem 4 can be obtained by substitution using the fact that the solution of a homogeneous Volterra equation of the second kind is trivial. Theorem 4 enables us to reduce the IBVP describing the continuous accretion of a viscoelastic ageing body (in general with discontinuous constitutive equations) to the problem (3.13), which has the same form as the IBVP describing the continuous accretion of an elastic body.

We observe that the equilibrium equation in the IBVP (3.13) involves new (as compared with (3.1)–(3.5)), even if familiar, terms containing loads concentrated in the domain occupied by the additional body ($f_1^0(x, t)$) and on the base surface ($f_2^0(x, t)$).

We shall now transform the IBVP (3.13) to a BVP for T^0, E and u. To do this we differentiate the first equation with respect to t. Then

$$
\nabla \cdot S + h - \theta(\tau^*(x)\,\tau_1^{-1} - 1)\, h_1 - \delta(\tau^*(x)\,\tau_1^{-1} - 1)\, h_2 = 0
\tag{3.15}
$$

$$
h = Rf, \quad h_1 = s_n^{-1} p^*(x) \frac{\partial}{\partial \tau} \left.\frac{\partial \omega(t,\tau)}{\partial t}\right|_{\tau = \tau^*(x)}, \quad h_2 = s_n^{-1} \tau_1^{-1} \int_{\tau_0}^{\tau_1} p_*(x,\tau) \frac{\partial}{\partial \tau} \frac{\partial \omega(t,\tau)}{\partial t}\, d\tau
\tag{3.16}
$$

with R acting on an arbitrary vector $a(x, t)$ by (see also (2.9))

$$
Ra(x,t) = \frac{1}{G(t)} \frac{\partial a(x,t)}{\partial t} + \int_{\tau_0(x)}^{t} \frac{\partial a(x,\tau)}{\partial \tau} \frac{\partial \omega(t,\tau)}{\partial t}\, d\tau + a(x, \tau_0(x)) \frac{\partial \omega(t, \tau_0(x))}{\partial t}
\tag{3.17}
$$

We also differentiate the boundary conditions on $S_i(t)$ ($i = 1, 2$) and the constitutive equation with respect to t. We have

$$
\begin{aligned}
&x \in S_1(t):\ n \cdot S = w_0 (w_0 = R p_0), \quad x \in S_2(t):\ \upsilon = \upsilon_0 \\
&S = 2D + (K-1)\, I_1(D)\,\mathbf{1}
\end{aligned}
\tag{3.18}
$$

To derive the boundary condition on $S^*(t)$ we apply the Hamilton operator to the initial-boundary condition $T^0(x, \tau^*(x)) = T^{*0}(x)$ as in [1, 3–5]. Then, after some reduction we find, using (3.9)–(3.13), that

$$
\begin{aligned}
&x \in S^*(t):\ n \cdot S = [\nabla \cdot T^*(x) + f^*(x)]\, G^{-1}(t)\, s_n - \theta(\tau^*(x)\,\tau_1^{-1} - 1)\, p^*(x) \times \\
&\times \left.\frac{\partial \omega(t,\tau)}{\partial \tau}\right|_{\tau = \tau^*(x)} - \delta(\tau^*(x)\,\tau_1^{-1} - 1)\, \tau_1^{-1} \int_{\tau_0}^{\tau_1} p_*(x,\tau) \frac{\partial}{\partial \tau}\left[\frac{1}{G(\tau)} + \omega(t,\tau)\right] d\tau \\
&n \cdot T^*(x) = p^*(x) \quad (f(x, \tau^*(x)) = f^*(x), t = \tau^*(x))
\end{aligned}
\tag{3.19}
$$

Now, supplementing (3.15), (3.18) and (3.19) with the Cauchy conditions in terms of rates from (3.13), we obtain a BVP for S, D and υ of the form

$$\begin{aligned}
&\nabla\cdot S+h-\theta(\tau^*(x)\tau_1^{-1}-1)h_1-\delta(\tau^*(x)\tau_1^{-1}-1)h_2=0\\
&x\in S_1(t):\ n\cdot S=w_0,\quad x\in S_2(t):\ \upsilon=\upsilon_0\\
&x\in S^*(t):\ n\cdot S=[\nabla\cdot T^*(x)G^{-1}(t)+f^*(x)G^{-1}(t)-\theta(\tau^*(x)\tau_1^{-1}-1)f_1^0(x,t)-\\
&\qquad -\delta(\tau^*(x)\tau_1^{-1}-1)f_2^0(x,t)]s_n,\quad n\cdot T^*(x)=p^*(x)\ (t=\tau^*(x))\\
&\qquad D=\tfrac{1}{2}[\nabla\upsilon+(\nabla\upsilon)^T],\quad S=2D+(K-1)I_1(D)\mathbf{1}
\end{aligned}\tag{3.20}$$

where f_1^0, f_2^0 and h, h_1, h_2 are defined by (3.12) and (3.16), (3.17), the conditions on $S_1(t)$ and $S^*(t)$ being identical.

Relations (3.20) supplemented with the initial conditions for the basic body at $t=\tau_1$, which also contain the initial-boundary condition on the growth surface, form an IBVP with t as a parameter.

Theorem 5. The functions T^0, E and u constitute a solution of the IBVP (3.13) if and only if S, D and υ satisfy (3.20) and ($\tau_1\leqslant t\leqslant\tau_2$)

$$T^0(x,t)=T^0(x,\tau^0(x))+\int_{\tau^0(x)}^t S(x,\tau)\,d\tau,\quad E(x,t)=E(x,\tau^0(x))+\int_{\tau^0(x)}^t D(x,\tau)\,d\tau \tag{3.21}$$

$$u(x,t)=u(x,\tau^0(x))+\int_{\tau^0(x)}^t \upsilon(x,\tau)\,d\tau$$

The values of stresses, strains and displacements at $t=\tau_1$ are naturally taken from the solution of the problem for the basic growing body already obtained.

Using a representation of the form (2.10) for the solution for the basic body, we can state one more relation between T^0, E, u and their rates, which holds for the whole growing body

$$T^0(x,t)=T^0(x,\tau_0(x))+\int_{\tau_0(x)}^t S(x,\tau)\,d\tau,\quad E(x,t)=E(x,\tau_0(x))+\int_{\tau_0(x)}^t D(x,\tau)\,d\tau \tag{3.22}$$

$$u(x,t)=u(x,\tau_0(x))+\int_{\tau_0(x)}^t \upsilon(x,\tau)\,d\tau$$

The values of the rates of the corresponding functions for the basic body over the time interval $[\tau_0, \tau_1]$ are known from the solution of (2.8).

Theorem 6. For T, E and u to be solutions of IBVP (3.1)–(3.5) it is necessary and sufficient that S, D and υ form a solution of (3.20) and that the following relations be satisfied (see (2.11))

$$\begin{aligned}
T(x,t)=G(t)\Bigg\{&\frac{T(x,\tau_0(x))}{G(\tau_0(x))}\left[1+\int_{\tau_0(x)}^t R(t,\tau)\,d\tau\right]+\\
&+\int_{\tau_0(x)}^t\left[S(x,\tau)+\int_{\tau_0(x)}^{\tau} S(x,\zeta)\,d\zeta\, R(t,\tau)\right]d\tau\Bigg\}
\end{aligned}\tag{3.23}$$

$$u(x,t)=u(x,\tau_0(x))+\int_{\tau_0(x)}^t \upsilon(x,\tau)\,d\tau$$

The proof of this theorem can be constructed using Theorems 4 and 5 as well as the representation (3.22).

It follows that the solution of the problem of the accretion of a viscoelastic ageing body over the time interval $[\tau_0, \tau_2]$ can be obtained by successive solution of the mathematically identical problems

(2.1)–(2.4) at $t = \tau_0$ and (2.8), (3.20) with t as a parameter, which have the same form as the BVP of the theory of elasticity. Then the true stresses and displacements in the growing body can be reconstructed from (3.23).

One can see that relations (3.23) are designed in such a way that the solutions of the formally transformed problem (2.8) can be used when the rates of the operator stresses S and displacement υ for the basic body are determined over the time interval $[\tau_0, \tau_1]$. Evaluation formulae equivalent to (3.23) can also be obtained from (2.6), (2.7) and (3.21). They are more complicated and not as universal as (3.23). They will therefore be omitted.

Relations (3.23) indicate that the SSS for a growing viscoelastic body depends on the whole history of loading and accretion of the body. The initial values of the displacements $u(x, \tau^*(x))$ of the deposited elements in (3.23) are usually set to be zero (since the SSS of the growing body does not depend on them).

4. THE DEFORMATION OF A VISCOELASTIC BODY AFTER ITS GROWTH HAS STOPPED

Suppose that the body ceases to grow at time τ_2. At this instant it occupies a domain Ω_1 with surface S_1, on which two kinds of boundary conditions are specified, as in the case of the problem for the basic body. Moreover $S^*(\tau_2) = S_1^* \subseteq \cup S_i(t)$ $(i = 1, 2)$. In this case over the time interval $[\tau_2, \tau_3]$ (τ_3 is the time the next stage of the growth process starts) the problem for the invariable body occupying body occupying Ω_1 is similar to (3.1)–(3.5) without the initial-boundary condition on $S^*(t)$

$$\nabla\cdot T + f = 0,\quad x\in S_1(t):\ n\cdot T = p_0,\quad x\in S_2(t):\ u = u_0 \tag{4.1}$$
$$D = \tfrac{1}{2}[\nabla\upsilon + (\nabla\upsilon)^T],\quad T = G(I + N(\tau_0(x), t))\,[2E + (K-1)\,I_1(E)\,\mathbf{1}]$$

with $\tau^*(x) = \tau_2$ for $x \in S_1^*$. The stresses, strains and displacements at $t = \tau_2$ found by solving the growth problem at the previous step serve as the initial conditions.

As before, one can obtain the following BVP (see (3.20))

$$\nabla\cdot S + h - \theta(\tau^*(x)\,\tau_1^{-1} - 1)\,h_1 - \delta(\tau^*(x)\,\tau_1^{-1} - 1)\,h_2 = 0$$
$$x\in S_1(t):\ n\cdot S = w_0,\quad x\in S_2(t):\ \upsilon = \upsilon_0 \tag{4.2}$$
$$D = \tfrac{1}{2}[\nabla\upsilon + (\nabla\upsilon)^T],\quad S = 2D + (K-1)\,I_1(D)\,\mathbf{1}$$

where h, h_1 and h_2 are defined by (3.16) and (3.17), the expression for w_i is given in (3.18), and the initial conditions remain as before.

One can prove the following theorem.

Theorem 7. For T, E and u to be solutions of the IBVP (4.1) it is necessary and sufficient that S, D and υ form a solution of (4.2) and that relations (3.23) be satisfied.

The proof of this theorem will be omitted. It is analogous to the argument in the previous section.

Thus, to construct the solution of the problem over the time interval $[\tau_0, \tau_3]$ one has to construct the solutions of the following four identical problems (having the same form as the BVP of the theory of elasticity with t as a parameter): problem (2.1)–(2.4) for $t = \tau_0$ as well as problems (2.8), (3.20) and (4.2). The SSS of the growing body can then be reconstructed for any $t \in [\tau_0, \tau_3]$ from (3.23).

5. PIECEWISE CONTINUOUS ACCRETION OF A VISCOELASTIC BODY AND BASIC CONCLUSIONS

Suppose that the growth process restarts at time τ_3 and deposition of new elements begin on the surface S_1 of the body (or part of it) occupying the domain Ω_1. Then, by analogy with Section 3, one can obtain a problem of the form (3.20) describing the behaviour of the growing viscoelastic body up to the instant of time τ_4 when the accretion stops again. Naturally, the new growth surface may not be related in any way to the previous one, the functions and parameters in (3.20) may take new values, and new known terms may appear in the equilibrium equations involving loads concentrated in the domains occupied by the sub-bodies and on the base surfaces subject to a load. Once the problem is solved, the SSS of the growing body can be determined using (3.23).

For $t \geqslant \tau_4$, when the body does not grow, the problem can be reduced in the same way to the form (4.2) and then (3.23) can be used.

The following step-wise scheme can be used to solve the problem of arbitrary piecewise-continuous accretion. First (2.1)–(2.4) for $t = \tau_0$ and (2.8) are solved. Then solutions of (3.2) and (4.2) are constructed at each stage involving either continuous accretion or, respectively, no growth at all. The final results can be obtained using (3.23).

Hence it follows that the process of piecewise-continuous accretion of a viscoelastic ageing body with any finite number of instants when the growth starts and stops can be considered using the method proposed. The problem with n instants when growth starts (and, naturally, n instants when it stops) can be reduced to the study of $2n + 2$ problems of one type, which have the same form as the BVP of the theory of elasticity containing t as a parameter (strictly speaking, the first problem for the basic body at $t = \tau_0$ is just the BVP of the theory of elasticity). Once these $2n + 2$ problems are solved, the SSS of the viscoelastic ageing body under consideration can be easily reconstructed for any time from the above formulae.

The one-to-one correspondence between the solutions of the problem of piecewise continuous accretion of a viscoelastic ageing body and the BVP of the theory of elasticity established in the present section enables us to conclude that a unique solution of the IBVP exists that describes the piecewise-continuous accretion of a viscoelastic body because a unique solution of the BVP of the theory of elasticity exists (see, for example, [13]).

We shall consider some characteristic ways in which solids can grow. We assume that the base surface of the growing body is free of any loads over the time intervals when no growth occurs, i.e. for $x \in S_* \subset S_1(\tau)$ we have $n \cdot T(x, \tau) = p_*(x, \tau) = 0$ (see (3.11)). Then one should set f_2^0 and h_2 to be zero in (3.20) (see (3.12) and (3.16)). Furthermore, let the growth surface itself be free of any loads during the growth process, i.e. let T^* be consistent with zero external forces. In this case $n \cdot T^*(x, \tau) = p^*(x) = 0$ (see (3.3)) and f_2^0 and h_1 are set to be zero in (3.20) (see (3.12) and (3.16)). The conditions mentioned above, which are natural in applied problems (see, for example, [4, 5, 14, 15]), result in a significant simplification of the analytic or numerical analysis of problems (3.20) and (4.3) for $t \geqslant \tau_1$, since the terms involving loads concentrated on the surface between the basic and additional bodies and in the domain of the additional body are missing.

We observe that even when there are constant volume forces acting on the body the resulting constitutive equations of the form (3.20) and (4.2) are equivalent to some BVP's of the theory of elasticity with t as a parameter containing volume forces inhomogeneous in the coordinates (see (3.16) and (3.17)).

Using the property of limited creep of a viscoelastic material (see [5, 7, 8]) one can obtain a number of other interesting results from (3.20), (3.12), (3.16) and (3.17). If one assumes that only the surface of the basic body is subject to a load, the actions are stationary, and accretion does not involve pretension, then the interaction between newly deposited particles and the body already formed can be neglected starting from some time t_0. In other words, starting from t_0 the growth process has little effect on the state of the part of the body formed prior to t_0 and the part formed for $t > t^0$ is practically stress-free. In particular, if the time when a stationary load is applied to the basic body is much earlier than the time when the accretion starts, all other conditions being equal, then the effect of accretion on the state of the basic body will be quite small and practically the whole additional body will be strain-free. Similar conclusions can be drawn when considering a load regime of the original body under which the actions remain constant for a prolonged period of time prior to the beginning of growth, irrespective of their variation at earlier times.

The effects considered have a clear mechanical meaning. Indeed, the deformation of a viscoelastic body will practically cease after a period of time under limited creep conditions and stationary actions. Subsequent deposition of stress-free elements leads to a situation when the interaction between the parts of the body already formed and those being created during the growth process is negligible.

Relations (2.8), (3.20), (3.23) and (4.2) also enable us to predict such phenomena inherent in growing bodies as the presence of residual stresses after the loads are removed, the presence of surfaces of stress discontinuity in the growing body, and the dependence of the SSS of a viscoelastic body on the growth rate (only the order of the acts of deposition and loading matters in the elastic case) (see [1–5]).

Finally, we shall discuss one more important aspect of the problem of accretion of a solid. It is concerned with the correspondence between the solution of the accretion IBVP and the viscoelasticity BVP for a variable boundary. The question is as follows: when will the solution of the non-classical accretion problem be the same as that of the classical problem of solid mechanics in a domain which varies with time? It turns out that the solutions are the same only when the strains in the growing body and the deposited elements can be made compatible [1–5]. Being a degenerate case of the IBVP describing the accretion of a solid, such an accretion regime clearly cannot be realized in practice.

Unlike the degenerate case when the strains in the whole body are compatible during the accretion process, in the case of stress-free elements being deposited on the body ($x \in S^*(t)$: $T = T^* = 0, n \cdot T^* = 0$ $(t = \tau^*(x))$), which is a completely relevant version of the accretion process, the problem will fail to become much simpler. It provides a brilliant demonstration of the effects related to accretion in model examples and is often encountered in applications. Here we have a situation when some inhomogeneous condition, rather than the homogeneous one, is trivial in a certain sense, unlike the traditional formulation of the BVP in solid mechanics.

This research was supported financially by the Russian Foundation for Basic Research (94-01-00181-a).

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Translated by T.J.Z.